Optimal Weighted Orthogonalization of Measured Modes

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A technique is described by which measured modes, which usually do not satisfy the theoretical requirement of weighted orthogonality, are forced to satisfy this condition in an optimal way. The corrected modes are closest to the measured ones in a weighted Euclidean sense. Direct and iterative methods for computing the corrected modes are shown and discussed. The stiffness matrix that complies with the required orthogonality conditions and is the closest matrix to a previously given stiffness matrix is also found.

Nomenclature

d	= weighted norm of the errors between the computed and optimal stiffness matrix
E	= orthogonality-error matrix
I	= unity matrix
K	= computed or measured stiffness matrix
$\tilde{K}$	= exact stiffness matrix
M	= mass matrix
m_{jk}	= jk element of M
N	= $M^{1/2}$
n_{ij}	= ij element of N
P	= modal matrix of T^tMT
p	= Nq
q	= general-coordinate vector
T	= measured mode shape matrix
$\tilde{T}_i$	= ith measured mode shape
T_i	= normalized $\tilde{T}_i$
t_{jk}	= jk element of T
$[\cdot]^t$	= transpose of $[\cdot]$
V	= NX
X	= corrected orthogonal mode shape matrix
$\tilde{X}$	= exact mode shape matrix
X_k	= kth iterated X
x_{jk}	= jk element of X
Y	= optimal stiffness matrix
y_{jk}	= jk element of Y
Γ	= $(TMT)^{-1/2} - I$
Γ_k	= kth iteration of Γ
δ_{il}	= Kronecker delta
η	= Lagrange function
$\beta_Y, \Lambda_T, \Lambda_Y$	= matrix of Lagrange multipliers
$\beta_{Yli}, \lambda_{Tli}, \lambda_{Yli}$	= li element of β_Y, Λ_T, and Λ_Y, respectively
Λ_{TK}	= kth iteration of Λ_T
λ_i	= ith eigenvalue of T^tMT
μ_i	= ith eigenvalue of E
ϕ	= weighted norm of $(X\text{-}T)$
ψ	= Lagrange function
Ω^2	= measured frequency matrix
$\tilde{\Omega}^2$	= exact frequency matrix
Ω^2_{il}	= il element of Ω^2

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I. Introduction

It is well known that mode shapes of a given structure that are determined experimentally by vibration tests are usually nonorthogonal. In order to be used in a dynamic analysis, these modes must be corrected to satisfy the orthogonality requirement. Several methods have been proposed in the literature for orthogonalization of measured mode shapes. Gravitz[1] proposed a method by which an asymmetric flexibility matrix is obtained from the measured modes. By averaging the off-diagonal terms, a symmetric flexibility matrix is obtained. This matrix is used to obtain new orthogonal modes together with altered frequencies. This method was modified by Rodden[2] to include rigid body modes. McGrew[3] proposed a Gram-Schmidt orthogonalization procedure that assigns a higher credibility to the measurements of lower-frequency modes. Berman and Flannelly[4] proposed a theory by which measured normal modes are used to modify an analytically derived incomplete model. Thoren[5] proposed to calculate the mass matrix, the stiffness matrix, and the damping matrix by which the measured mode shapes satisfy the required orthogonality conditions. Collins et al.[6] proposed a statistical method by which experimental measurements of the natural frequencies and mode shapes are used to modify stiffness and mass characteristics of a finite element model. Berman[7] used a modified version of the incomplete model method to obtain dynamic equations of motion of a helicopter transmission gearbox from shake test data. Recently, Targoff[8] introduced intuitively an orthogonalization procedure to correct the symmetric errors.

In the present work, an entirely different approach is introduced that seeks an orthogonalization procedure which, in a certain sense, yields the optimally corrected modes. As one of the results of this approach, we show that the technique proposed by Targoff is an optimal one in that sense. Moreover, in the context of optimality, not all the modes of the model have to be measured. The approach of this work is based on work done in inertial navigation.[9-15] Basically, the method consisted of minimization of a Euclidean norm of the errors subject to the orthogonality requirement. Now, this method has been modified to include the treatment of weighted orthonormal conditions and number of mode shapes less than the number of the degrees of freedom of the structure.

The corrected mode shapes are further used to complete a corrected stiffness matrix that is obtained by another minimization of an Euclidean norm of a certain error matrix. If now the corrected stiffness matrix is used to compute the mode shapes and their corresponding frequencies, the latter will include the measured frequencies and the corrected modes. Hopefully, the additional mode shapes and their frequencies will be closer to the real not measured quantities.

II. Closed-Form Solutions

A. Optimal Orthogonalization of the Modal Matrix

The problem is now cast in the following mathematical form: Given a measured matrix $T(n \times m)$, $(n \geq m)$, a symmetric positive definite mass matrix $M(n \times n)$, find a matrix X that minimizes the weighted Euclidean norm

$$\phi = \| N(X-T) \| = \sum_{i=1}^{n} \sum_{k=1}^{m} \left[\sum_{j=1}^{n} n_{ij} (x_{jk} - t_{jk}) \right]^2 \tag{1}$$

$$N = M^{1/2} \tag{2}$$

and satisfies the weighted orthogonality condition

$$X^t M X = I \tag{3}$$

It must be noted that the measured modes T_i have to be normalized in the following way

$$T_i = \tilde{T}_i (\tilde{T}_i^t M \tilde{T}_i)^{-1/2} \tag{4}$$

where $\tilde{T}_i$ is the mode shape before the normalization. In these developments, it is assumed that M is known with a very high confidence. This assumption is ususally made in the literature.

Some explanation is needed to justify the use of the norm (1). Following Eq. (3), a new matrix can be defined

$$V = NX \tag{5}$$

Substitution of Eq. (5) into Eq. (3) yields

$$V^T V = I \tag{6}$$

It seems that Eqs. (5) and (6) justify the use of the weighted Euclidean norm (1).

Using Lagrange multipliers to incorporate the constraint of Eq. (3) into the cost function, the following Lagrange function is defined

$$\psi = \phi + \| \Lambda_T (X^t M X - I) \| \tag{7}$$

where Λ_T is a matrix of Lagrange multipliers and

$$\| \Lambda_T (X^t M X - I) \| \triangleq \sum_{l=1}^{m} \sum_{i=1}^{m} \lambda_{Tli} \left(\sum_{j=1}^{n} \sum_{k=1}^{n} x_{ji} m_{jk} x_{kl} - \delta_{il} \right) \tag{8}$$

δ_{il} is the Kronecker delta. It can be shown that Eq. (8) is the proper introduction of the $n \times n$ scalar constraint equations implied by Eq. (3). It is noted that due to the symmetry of Eq. (3) Λ_T must be symmetric for the problem to be well defined:

$$\Lambda_T = \Lambda_T^t \tag{9}$$

The partial differentiation of Eq. (7) with respect to x_{ij}, where the results are equated to zero, yields equations that x_{ij} have to satisfy when ψ is minimal. These equations are expressed in matrix form as follows (for more details, see Appendix A of Ref. 16).

$$\frac{\partial \psi}{\partial X} = 2M(X-T) + 2MX\Lambda_T = 0 \tag{10}$$

and, using the fact that M is invertible, Eq. (10) yields

$$X[I + \Lambda_T] = T \tag{11}$$

Equation (11) reveals the physical significance of the Lagrange multiplier matrix Λ_T. It represents the error between the measured and required mode shapes. Now, it is supposed that the error in the measured modes is small and therefore the matrix $[I + \Lambda_T]$ can be inverted thus

$$X = T[I + \Lambda_T]^{-1} \tag{12}$$

Substitution of Eq. (12) into Eq. (3) yields

$$I + \Lambda_T = (T^t M T)^{1/2} \tag{13}$$

Again, one can see that the error matrix Λ_T represents the deviation from the weighted orthonormality of the measured mode shapes.

Substitution of Eq. (13) into Eq. (12) yields the finite result

$$X = T(T^t M T)^{-1/2} \tag{14}$$

Appendix A of Ref. 16 introduces a different approach, in which a directional derivative of a function is applied to obtain Eq. (14) from Eq. (7). Using that approach it is shown there that Eq. (14) indeed yields the minimum of ψ provided that the positive square root of $T^t M T$ is used in Eq. (14).

Clearly, the mode shapes, X, defined by Eq. (10) satisfy the weighted orthonormality condition (3) and are the closest, in the weighted Euclidean sense, to the measured mode shapes T.

Targoff quite intuitively obtained Eq. (14), although not explicitly presenting this result. He correctly maintained that any correction to T that is based on the orthogonality test matrix $T^t M T$ can correct only symmetric errors. He then made the a priori assumption that the orthogonality errors are indeed symmetric, which enabled him to compute a symmetric matrix α (which is identical to Λ_T) and then, using an expression like Eq. (12), compute X. The search for an optimal X in the sense presented here automatically precludes asymmetric errors. This point is well explained by Mortensen[17] and also briefly discussed by Targoff.[8]

B. Optimal Correction of the Stiffness Matrix

Having obtained the orthonormalized modes, we shall now use them to compute an optimally corrected stiffness matrix. Again we shall assume that M is positive definite and is known quite accurately; hence M is considered to be an exact matrix. The relationship between M and the exact symmetric stiffness matrix $\tilde{K}$ is given by the basic vibration equation

$$M\ddot{q} + \tilde{K} q = 0 \tag{15}$$

where $\tilde{K}$ may be singular to include the rigid-body motions.

Following Eq. (5), a new vector p is defined by

$$p = Nq \tag{16}$$

Substitution of Eq. (16) into Eq. (15) and using the fact that N is invertible, yields

$$\ddot{p} + N^{-1} \tilde{K} N^{-1} p = 0 \tag{17}$$

The governing matrix in Eq. (16) is the symmetric matrix, $N^{-1} \tilde{K} N^{-1}$, whose Euclidean norm, due to transformation (16), is consistent with the weighted norm of the modal matrix given in Eq. (1). It appears that the Euclidean norm of $N^{-1} \tilde{K} N^{-1}$ is the natural weighted norm of the stiffness matrix.

Let $\tilde{X}$ denote the exact orthonormal modal matrix. Then, M, $\tilde{K}$, and $\tilde{X}$ are related by

$$\tilde{X}^t M \tilde{X} = I$$

$$\tilde{K}\tilde{X} = M\tilde{X}\tilde{\Omega}^2 \tag{18}$$

where the diagonal matrix $\tilde{\Omega}^2$ represents the exact natural frequencies.

It is widely accepted that the measured frequencies constitute the most accurate test data. Hence, we take the measured frequencies Ω^2 as our best estimate of the exact frequencies $\tilde{\Omega}^2$. Since $\tilde{X}$ is also unknown we use X, the corrected mode shape matrix, as our best estimate of $\tilde{X}$ for which we seek a symmetric stiffness matrix Y, such that the constraints[18] (18),

$$YX = MX\Omega^2, \qquad Y = Y^t \tag{19}$$

will be satisfied. Now Eqs. (19) do not fully define Y; therefore, we shall seek that Y which is closest to a given measured or computed stiffness matrix K. In view of Eq. (17), the most natural way to measure the closeness of Y to K is by evaluating d, which is given by

$$d = \tfrac{1}{2} \| N^{-1}(Y-K)N^{-1} \| \tag{20}$$

In summary, we seek now that stiffness matrix Y which minimizes d subject to the constraint expressed by Eqs. (19). Using Lagrange multipliers to incorporate the constraints of Eqs. (19) into the new Lagrange function, we use two matrices of Lagrange multipliers Λ_Y and β_Y to define the Lagrange function η as follows

$$\eta = d + 2\|\Lambda_Y(YX - MX\Omega^2)\| + \|\beta_Y(Y - Y^t)\| \tag{21}$$

where

$$2\|\Lambda_Y(YX - MX\Omega^2)\|$$

$$= 2\sum_{k=1}^{n}\sum_{p=1}^{m} \lambda_{Ykp}\left(\sum_{l=1}^{n} y_{kl}x_{lp} - \sum_{l=1}^{n} m_{kl}\sum_{q=1}^{m} x_{lq}\Omega^2_{qp}\right) \tag{22a}$$

and

$$\|\beta_Y(Y-Y^t)\| = \sum_{k=1}^{n}\sum_{l=1}^{n}\beta_{Ykl}(y_{kl} - y_{lk}) \tag{22b}$$

Λ_Y is a matrix of order $(n\times m)$, β_Y is an antisymmetric matrix of order $(n\times m)$:

$$\beta^t_Y = -\beta_Y \tag{23}$$

and

$$\Omega^2_{pq} = \Omega^2_p \text{ for } p=q, \quad \Omega^2_{pq} = 0 \text{ for } p \neq q$$

where Ω_p are the measured frequencies.

The partial differentiation of η with respect to y_{ij} and equating the results to zero yield equations that y_{ij} must satisfy when η is minimal. In matrix form, they are

$$\frac{\partial\eta}{\partial Y} = M^{-1}(Y-K)M^{-1} + 2\Lambda_Y X^t + 2\beta_Y = 0 \tag{24}$$

Taking Eq. (23) into account β_Y can be easily eliminated to obtain

$$Y = K - M\Lambda_Y X^t M - MX\Lambda^t_Y M \tag{25}$$

Using Eqs. (3) and (19), one obtains

$$MX\Omega^2 = KX - M\Lambda_Y - MX\Lambda^t_Y MX \tag{26}$$

It can be easily verified that Eq. (26) is satisfied by the following Λ_Y,

$$\Lambda_Y = M^{-1}KX - \tfrac{1}{2}XX^tKX - \tfrac{1}{2}X\Omega^2 \tag{27}$$

Substitution of Eq. (27) into Eq. (25) yields

$$Y = K - KXX^tM - MXX^tK + MXX^tKXX^tM + MX\Omega^2X^tM \tag{28}$$

In Ref. 18 it is shown that Y is unique and therefore minimizes the positive function d (Eq. 20). It is noted that when X contains the full set of mode shapes it is invertible and then using Eq. (3) it can be shown that

$$MXX^t = X^tMX = XX^tM = I \tag{29}$$

which when substituted into Eq. (28) yields

$$Y_{\text{full}} = MX_{\text{full}}\Omega^2_{\text{full}}X^t_{\text{full}}M \tag{30}$$

III. Direct Solution

To obtain the optimally orthogonalized mode shapes, one has to solve Eq. (14) where $(T^tMT)^{-1/2}$ has to be computed. A straightforward computation of this inverse root requires finding the eigenvalues of T^tMT which amounts to solving an mth order polynomial. Now, since in practice most, if not all, the mode shapes are nearly orthogonal, T^tMT is almost a unit matrix. This implies that most, if not all, the eigenvalues are close to unity, which may cause difficulties in solving that polynomial. This difficulty can be avoided in the following way. Define an error matrix E where

$$E = T^tMT - I \tag{31}$$

Let λ_i denote the eigenvalues of T^tMT and μ_i the eigenvalues of E then it can be easily shown that

$$\lambda_i = 1 + \mu_i \tag{32}$$

and the respective eigenvectors of T^tMT and E are equal.

The relative distance between the various μ_i is larger than that of λ_i, therefore it is easier to find μ_i than to find λ_i. With μ_i on hand, it is possible to obtain $(T^tMT)^{-1/2}$ by

$$(T^tMT)^{-1/2} = PL^{-1/2}P^t = P\left[\diagdown \frac{1}{+\sqrt{\lambda_i}} \diagdown\right]P^t = P\left[\diagdown \frac{1}{+\sqrt{1+\mu_i}} \diagdown\right]P^t \tag{33}$$

where P is an orthonormal matrix whose columns are the eigenvectors of E (and also of T^tMT).

As was mentioned previously the fact that only the positive square root has to be used is explained in Appendix A of Ref. 16. Here, we shall examine a boundary case where it is assumed that the measured modes happen to be orthogonal, that is

$$T^tMT = I \tag{34}$$

and the obvious solution for X is

$$X = T \tag{35}$$

In this case, P is equal to I and λ_i are all equal to unity. From Eqs. (33) and (14) it is obvious that to obtain the correct answers, only the positive root of λ_i has to be used in Eq. (33). It could be argued that for continuity reasons this sign choice should not be altered in cases where the modes are not quite orthogonal.

IV. Iterative Solutions

The problem of computing $(T^tMT)^{-1/2}$ can be circumvented by using iterative computational processes to compute Eq. (14). Three such processes, for the case where $M=I$, were introduced in Ref. 14. The first one was named the original process. Its modified version to include $M \neq I$ is as follows

$$X_{k+1} = 3/2X_k - \tfrac{1}{2}X_kX^t_kMX_k \tag{36}$$

$$X_0 = T$$

Table 1 Element of the mass matrix (kg)

M(1, 1)=0.175	M(11,11)=0.333	M(21,21)=0.454	M(31,31)=0.103
M(2, 2)=0.030	M(12,12)=0.290	M(22,22)=0.508	M(32,32)=0.400
M(3, 3)=0.020	M(13,13)=0.607	M(23,23)=1.362	M(33,33)=0.425
M(4, 4)=0.030	M(14,14)=0.170	M(24,24)=0.250	M(34,34)=0.470
M(5, 5)=0.060	M(15,15)=0.467	M(25,25)=0.150	M(35,35)=1.147
M(6, 6)=0.264	M(16,16)=0.428	M(26,26)=0.314	M(36,36)=0.321
M(7, 7)=0.140	M(17,17)=0.250	M(27,27)=0.633	M(37,37)=0.150
M(8, 8)=0.180	M(18,18)=1.156	M(28,28)=0.698	
M(9, 9)=0.116	M(19,19)=0.253	M(29,29)=1.290	
M(10,10)=0.233	M(20,20)=0.221	M(30,30)=0.055	

Table 2 Matrix T^tMT

0.10000E 01	-0.95554E-01	-0.26906E-01	0.27638E 00	-0.26139E 00
-0.95554E-01	0.10000E 01	-0.98462E-01	0.77953E-01	0.28041E 00
-0.26906E-01	-0.98462E-01	0.10000E 01	-0.59210E-01	-0.42166E-01
0.27639E 00	0.77954E-01	-0.59210E-01	0.10000E 01	-0.34041E-02
-0.26139E 00	0.28041E 00	-0.42166E-01	-0.34040E-02	0.10000E 01

Table 3 Five normalized measured modes ($kg^{-1/2}$)

0.49803E 00	0.23278E 00	0.61370E 00	-0.73181E 00	-0.87305E 00
0.67913E 00	0.49156E 00	0.54005E 00	-0.34283E 00	-0.52383E 00
0.79685E 00	0.82577E 00	0.35594E 00	0.18021E 00	-0.52383E 00
0.86023E 00	0.11210E 01	0.13501E 00	0.14768E 01	-0.21826E 00
0.90551E 00	0.13004E 01	0.35594E-01	0.21976E 01	-0.29684E 00
0.27165E 00	-0.14435E 00	0.54496E 00	-0.30327E 00	-0.44525E 00
0.56141E 00	0.32510E-01	0.21909E 00	-0.30767E 00	-0.15715E 00
0.64291E 00	0.17296E 00	-0.98191E-02	-0.65929E-01	0.52383E-01
0.73346E 00	0.68272E 00	-0.54373E 00	0.10131E 01	0.91670E 00
0.10232E 01	-0.26008E 00	0.50753E 00	0.11208E 00	-0.20953E 00
0.29882E 00	-0.19506E 00	0.21357E 00	0.15383E-01	-0.19207E 00
0.36220E 00	-0.13004E 01	0.34981E-01	-0.30767E-01	-0.87305E-01
0.41653E 00	0.46815E-01	-0.11660E 00	-0.39557E-01	0.22699E-01
0.58858E 00	0.14045E 00	-0.85917E 00	0.17801E 00	0.35795E 00
0.22638E-01	-0.15345E 00	0.31298E 00	0.28349E 00	-0.17461E-01
0.14035E 00	-0.17556E 00	0.27616E 00	0.30767E 00	-0.61113E-01
0.17205E 00	-0.17556E 00	0.15956E 00	0.21976E 00	-0.13969E 00
0.27165E 00	-0.14305E 00	-0.12949E 00	0.76917E-01	-0.15715E 00
0.43283E 00	-0.55918E-01	-0.68488E 00	0.15383E-01	0.11350E 00
0.16299E-01	-0.58519E-01	0.24916E 00	0.23075E 00	0.69844E-01
0.23543E-01	-0.65021E-01	0.26143E 00	0.31866E 00	0.52383E-01
0.34409E-01	-0.87128E-01	0.14483E 00	0.27470E 00	-0.68098E-01
0.63385E-01	-0.10403E 00	-0.55233E-01	0.17142E 00	-0.27938E 00
0.10051E 00	-0.11444E 00	-0.54558E 00	0.13186E 00	-0.41906E 00
0.27165E 00	-0.10403E 00	-0.74134E 00	0.54941E-01	-0.17461E 00
-0.24449E-01	-0.18206E 00	0.14422E 00	0.14944E 00	0.13096E-01
0.28976E-01	-0.13004E-01	0.19270E 00	0.21976E 00	0.96035E-01
0.43464E-01	-0.26008E-01	0.10494E 00	0.20218E 00	-0.69844E-02
0.59763E-01	-0.33811E 00	-0.12888E-01	0.15823E 00	-0.22699E 00
0.86929E-01	-0.42914E-01	-0.20682E 00	0.15383E 00	-0.57621E 00
0.86929E-01	-0.70223E-01	-0.70575E 00	0.14724E 00	-0.10477E 01
-0.36220E-01	-0.78025E-02	0.46027E-01	0.61534E-01	-0.43652E-01
0.21732E-01	0.00000E 00	0.14729E 00	0.13186E 00	0.52383E-01
0.22638E-01	-0.39013E-02	0.92054E-01	0.12966E 00	0.17461E-01
0.43464E-01	-0.91029E-02	0.00000E 00	0.11867E 00	-0.19207E 00
0.70630E-01	-0.13004E-01	-0.12888E 00	0.11647E 00	-0.49764E 00
0.81496E-01	-0.23408E-01	-0.53698E 00	0.13186E 00	-0.89487E 00

In Appendix C of Ref. 16, it is shown that when the largest eigenvalue of T^tMT is less than $\sqrt{3}$ the process converges to X. All eigenvalues must be nonzero.

The second process, called the dual process, requires an inversion of a matrix. However, in our case the number of measured modes is less than the number of the degrees of freedom of the system; consequently, this matrix is a singular one. To obtain the iterative process, therefore, the pseudo-inverse rather than the inverse is used here. This yields the following analogous process, which we name the modified dual process

$$X_{k+1} = \tfrac{1}{2} X_k [I + (X_k^t M X_k)^{-I}] \tag{37}$$

$$X_0 = T$$

Note that in the iterative process itself there appears only a regular matrix inverse. The process always converges whenever T^tMT is regular.

Table 4 Orthogonalized mode shapes (kg $^{-1/2}$)

0.53809E 00	0.44457E 00	0.60471E 00	−0.81491E 00	−0.86563E 00
0.72646E 00	0.64847E 00	0.55982E 00	−0.45981E 00	−0.51608E 00
0.77976E 00	0.96663E 00	0.40462E 00	0.44069E −01	−0.55932E 00
0.69679E 00	0.11615E 01	0.23288E 00	0.13532E 01	−0.29586E 00
0.62800E 00	0.13224E 01	0.15818E 00	0.20838E 01	−0.41262E 00
0.27700E 00	−0.38324E −01	0.53132E 00	−0.32802E 00	−0.39971E 00
0.62630E 00	0.99380E −01	0.22030E 00	−0.39835E 00	−0.84785E −01
0.70597E 00	0.19645E 00	0.67973E −02	−0.17623E 00	0.12047E 00
0.76479E 00	0.53158E 00	−0.46633E 00	0.87566E 00	0.94766E 00
0.10396E 01	−0.18761E 00	0.51281E 00	−0.12838E −01	−0.38619E −01
0.28521E 00	−0.15611E 00	0.20760E 00	−0.12243E −01	−0.13158E 00
0.33419E 00	−0.13228E 01	−0.21905E −01	−0.24621E −01	0.14264E 00
0.45154E 00	0.55851E −01	−0.10953E 00	−0.11033E 00	0.73741E −01
0.63927E 00	0.66826E −01	−0.83896E 00	0.60358E −01	0.42404E 00
−0.22436E −01	−0.15475E 00	0.31440E 00	0.30525E 00	0.57781E −02
0.91605E −01	−0.17040E 00	0.27804E 00	0.31315E 00	−0.22030E −01
0.12623E 00	−0.15954E 00	0.15804E 00	0.21582E 00	−0.10087E 00
0.24841E 00	−0.12674E 00	−0.13323E 00	0.44361E −01	−0.11184E 00
0.46109E 00	−0.92076E −01	−0.68265E 00	−0.66475E −01	0.17732E 00
−0.67979E −02	−0.69257E −01	0.25455E 00	0.24404E 00	0.83836E −01
−0.15039E −01	−0.76809E −01	0.26856E 00	0.33492E 00	0.65873E −01
−0.15794E −01	−0.85720E −01	0.14768E 00	0.28811E 00	−0.57825E −01
−0.17535E −02	−0.76886E −01	−0.59338E −01	0.17643E 00	−0.27594E 00
0.16960E −01	−0.87601E −01	−0.55515E 00	0.12094E 00	−0.42349E 00
0.24169E 00	−0.11115E 00	−0.74724E 00	0.53421E −02	−0.14431E 00
−0.51118E −01	−0.19198E 00	0.13995E 00	0.17029E 00	0.35642E −01
0.12662E −01	−0.27385E −01	0.20022E 00	0.22676E 00	0.10622E 00
0.14841E −01	−0.29079E −01	0.10969E 00	0.20667E 00	0.14067E −03
−0.36251E −02	−0.32492E 00	−0.27255E −01	0.17428E 00	−0.18705E 00
−0.15077E −01	0.23166E −01	−0.21325E 00	0.15348E 00	−0.59718E 00
−0.87224E −01	0.38671E −01	−0.72305E 00	0.14442E 00	−0.10991E 01
−0.53517E −01	−0.38659E −02	0.46221E −01	0.71578E −01	−0.50496E −01
0.12222E −01	−0.59704E −02	0.15231E 00	0.13589E 00	0.58145E −01
0.77703E −02	−0.75112E −02	0.96064E −01	0.13295E 00	0.21125E −01
0.12088E −02	0.13681E −01	0.29150E −03	0.12018E 00	−0.19802E 00
−0.13686E −01	0.47911E −01	−0.13352E 00	0.11634E 00	−0.51874E 00
−0.65542E −01	0.74029E −01	−0.54947E 00	0.12844E 00	−0.94211E 00

Table 5 Matrix X^tMX

0.10000E 01	−0.30035E −07	0.37253E −07	0.23283E −08	0.48429E −07
−0.25844E −07	0.10000E 01	0.40978E −07	0.36089E −07	0.37253E −07
0.33528E −07	0.48429E −07	0.10000E 01	0.37253E −08	−0.11921E −06
−0.57742E −07	0.35623E −07	−0.74506E −08	0.10000E 01	0.59605E −07
0.78231E −07	0.44703E −07	−0.59605E −07	0.52154E −07	0.10000E 01

The third process, which was named the gradient projection process, is modified too to include $M \neq I$, and is as follows

$$X_{k+1} = X_k - \tfrac{1}{2}(X_k T^t M X_k - T) \qquad (38)$$

$$X_0 = T$$

It is shown in Appendix C of Ref. 16 that the Original and the Modified Dual Processes are second-order processes, whereas the Gradient Projection is a linear process. However, theoretically, the latter process possesses an attractive feature of convergence to the exact solution when it converges. Empirically though, it was found that the other processes are no less accurate. The eigenvalues of T^tMT must nonzero and less than $\sqrt{2}$ for the process to converge (see Appendix C of Ref. 16). The range and rate of convergence of the above three processes are discussed in detail in Appendix C of Ref. 16.

Targoff[8] implies the following iterative computation. Rewrite Eq. (12)

$$X = T(I + \Lambda_T)^{-1} \qquad (39)$$

where Λ_T can be computed using the following process

$$\Lambda_{T,k+1} = \tfrac{1}{2}(T^t MT - I)(I + \tfrac{1}{2}\Lambda_{T,k})^{-1} \qquad \Lambda_{T,0} = 0 \qquad (40)$$

We propose a different process that does not require an inversion of a matrix at every iteration as is the case when Eq. (40) is used. It is based on Eq. (41). Suppose that

$$X = T(T^t MT)^{-1/2} = T(I + \Gamma) \qquad (41)$$

where Γ is a correction matrix. From Eq. (41) one obtains

$$\Gamma = \tfrac{1}{2}[(T^t MT)^{-1} - I - \Gamma^2] \qquad (42)$$

$$\Gamma_{k+1} = \tfrac{1}{2}[(T^t MT)^{-1} - I - \Gamma_k^2] \qquad (43)$$

$$\Gamma_0 = 0$$

It is shown in Appendix D of Ref. 16 that the latter is a linear process and that a sufficient condition for convergence is that all the eigenvalues of T^tMT are larger than 0.25.

A slightly different iteterative process with features similar to the previous one can be obtained from Eqs. (12) and (14),

$$X = T(T^t MT)^{-1/2} = T(I + \Lambda_T)^{-1} \qquad (44)$$

which yields

$$\Lambda_{T,k+1} = \tfrac{1}{2}(T^t MT - I - \Lambda_{T,k}^2) \qquad \Lambda_{T,0} = 0 \qquad (45)$$

A sufficient condition for the convergence of Eq. (45) is that all the eigenvalues of $T^t MT$ be nonzero and less than 4.0.

V. An Example

The structure of a lifting surface of a modern flight vehicle was modeled by a 37 degree of freedom lumped model. The mass matrix of this model was computed as a diagonal matrix whose values are given in Table 1. The measured frequencies were 34.9, 94.3, 133.8, 174.7, and 226.5 Hz, respectively. Only five modes were measured. The matrix $T^t MT$ of these modes is given in Table 2. The measured modes are given in Table 3. The modes were orthogonalized and the resulting mode shapes are given in Table 4. The matrix $X^t MX$ is given in Table 5. The stiffness matrix, K, computed by the finite element method can be corrected using Eq. (28) to yield Y. This last matrix can be used directly in the strength and aeroelastic calculations of the lifting surface or, in case of high discrepancies, to check the final element model.

VI. Conclusions

An optimal orthogonalization of measured mode shapes was presented. It was based on the usual assumption that the mass matrix is accurately known. A closed-form solution as well as iterative solutions were given. The orthogonalization does not require that all the mode shapes be measured. Assuming that the measured modal frequencies are accurate and using the orthogonalized modes, it was shown how to optimally correct the computed stiffness matrix.

Finally it should be noted that engineering judgment always has to be used rather than blind reliance on computer programs to yield automatically the optimally computed modal or stiffness matrix since neither faulty measurements nor wrong calculations can be compensated by the techniques presented here. The latter are useful only for improving the accuracy in cases where the measurements and the computations are basically sound.

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